

Model of the p -Adic Random Walk in a Potential

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October 13, 2015

Abstract

We consider the p -adic random walk model in a potential, which can be viewed as a generalization of p -adic random walk models used for description of protein conformational dynamics. This model is based on the Kolmogorov–Feller equations for the distribution function defined on the field of p -adic numbers in which the probability of transitions per unit time depends on ultrametric distance between the transition points as well as on function of potential violating the spatial homogeneity of a random process. This equation, which will be called the equation of p -adic random walk in a potential, is equivalent to the equation of p -adic random walk with modified measure and reaction source. With a special choice of a power-law potential the last equation is shown to have an exact analytic solution. We find the analytic solution of the Cauchy problem for such equation with an initial condition, whose support lies in the ring of integer p -adic numbers. We also examine the asymptotic behaviour of the distribution function for large times. It is shown that in the limit $t \rightarrow \infty$ the distribution function tends to the equilibrium solution according to the law, which is bounded from above and below by power laws with the same exponent. Our principal conclusion is that the introduction of a potential in the ultrametric model of conformational dynamics of protein conserves the power-law behaviour of relaxation curves for large times.

1 Introduction

An ultrametric space is a metric space M with metric $d(x_1, x_2)$, $x_1, x_2 \in M$ satisfying the strong triangle inequality

$$\forall x_1, x_2, x_3 \in M : d(x_1, x_2) \leq \max\{d(x_1, x_3), d(x_2, x_3)\}. \quad (1)$$

A metric $d(x_1, x_2)$ satisfying (1) is called an ultrametric. A classical example of an ultrametric space is the field of p -adic numbers \mathbb{Q}_p [1, 2, 3]. The p -adic numbers were introduced in 1889 by K. Hesel and were proved widely useful in algebraic geometry, theory of numbers and representation theory.

For a long time the ultrametric approach have been found useful in solving various problems in the field of classification of objects and information processing of data arrays [4]. In the last 30 years the ultrametric analysis had had a recent reemergence—it received a great impetus from the pioneering works of researches from the scientific school of Academician V. S. Vladimirov, whose efforts were later taken up by a number of researches from different scientific schools (for an overview, consult, for example, [5]). This relative new research field is now known as the “ p -adic and ultrametric mathematical physics” and is blessed by a number of books and an immense number of research articles in the area of p -adic analysis, p -adic mathematical physics and their applications to modeling in various areas of physics, biology, computer science, sociology, physiology, etc. (see, for example, [5, 6] and the references given therein).

First ultrametric models in physics emerged first in the 1970s in the theory of spin systems with disorder [4, 7, 8, 9]. It was found that if a system has a large number of multiscale “internal contradictions” (frustrations), then it may reach an equilibrium in hierarchically nested regions of the phase space, the nesting level increases with decreasing temperature. Here, the ratio of scales of phase domains satisfies the strong triangle inequality, and thus the low-temperature spin states are found to be ultrametrically correlated. Almost immediately

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after the emergence of the ultrametric model of spin glass, a conjecture was made about the ultrametricity of the space of conformational states of a protein molecule [10, 11]. It should be noted, however, that a literal transfer of the ultrametric description of phase states of the spin glass and conformational states of protein molecule was triggered by a then-popular analogy between the low-temperature proteins and glasses [12]. As in the case of glasses, the atomic mobility of polymer globules is accompanied, in particular, by multiscale constraints generating highly intersected energy landscapes. In this sense, proteins resemble glasses, some of their thermodynamical properties are the same (for example, the low-temperature behaviour of heat capacities). But, as distinct from disordered globular structures, proteins have a unique special feature—they are capable of performing precise operations with objects of atomic scale, the physical reasons behind it remaining unclear.

A systematic treatment of the conformational dynamics of proteins based on an ultrametric model is given in a series of papers by the authors [13, 14, 15, 16, 17, 18, 19, 20, 21]. In these papers, the conformational dynamics of protein was described as follows. The conformational state of a protein was considered as a quasi-equilibrium macrostate, which includes many microstates of a macromolecule. A conformational state which is understood in exactly this sense is also called a “basin”. Further, there is a great number of bonds on mechanical degrees of freedom of atoms in a macromolecule. As a result, there are many constraints on the flexibility of a macromolecule at various scales of the configuration space of degrees of freedom of the entire macromolecule. It is assumed that a random walk of a protein over the configuration space is effected as follows. The most probable are such motions of atoms in a macromolecule which take place in small regions of the configuration space (small-scale rearrangements of atoms). In order that there would be a rearrangement of atoms on a large scale involving a much larger region of the configuration space, it is necessary that rearrangements of smaller scales would result in a configuration of a macromolecule from which a passage to a different region of the configuration space would be enabled by the available bonds. The probability of such an event is much smaller than the transition probability between arbitrary small-scale configurations. Hence, the characteristic times of probability transitions between configurations are much smaller on more fine scales than those of probability transitions between configurations on more coarse scales of the configuration space of a macromolecule. In accordance with this approach, one may conventionally single out an increasing family of scales (characteristic sizes of the regions) of the configuration space of a macromolecule. Correspondingly, the characteristic times of a macromolecule to stay in regions of the configuration space of various scales increase with increasing scale. From such a model of a random walk of a protein over regions of the configuration space (conformations or basins) of various scales one may catch the hierarchy of transitions and the hierarchy of characteristic times of the transitions. The indexation of such a hierarchy enables one to introduce in a natural way an ultrametric distance between conformational states. This model features a number of simplifying assumptions. Namely, it is assumed that the distance graph is a Kelley tree and that all conformations are homogeneous in energy. These assumptions are fairly restrictive, but nevertheless, they enable one to formalize such a model by a p -adic random process. In doing so it is assumed that the probabilities of transitions per unit time through the activation barriers (that is, transitions between conformational macrostates) obey the Arrhenius relation. Such a model, which looks simple at first sight, has enabled one to adequately describe two principal experiments: the experiment on spectral diffusion in proteins [17] and the experiment on binding kinetics of *CO* by myoglobin [20, 21]. We note that in these two experiments, the fluctuation dynamics of a protein is described in a unified way.

The further development of this model consists in its generalization by cancelling one or several simplifying assumptions. One of the ways towards it is to reject the assumption on the homogeneity of conformations relative to energies. This can be achieved by introducing a potential in the model.

The statement of the problem of ultrametric diffusion in a potential well is discussed in the literature on p -adic mathematical physics starting from the 2000s. Here, there is a principal difficulty regarding the introduction of a potential in the equation of an ultrametric random walk. It is known (see, for example, [22]) that the right-hand side of the equation for the distribution function of an arbitrary time-homogeneous Markov process contains three terms. The first term describes the drift of a trajectory and is related with the potential force, the second term describes the diffusion processes, and the third term describes the jump processes. In the case when the trajectories of a random process are everywhere continuous, the term responsible for the jump processes vanishes and the equation becomes the Fokker–Planck equation, which contains two terms—the term responsible for the potential force acting on a walking particle and the diffusion term. Since a p -adic random walk is supported by functions that are constant everywhere except at discontinuities of the first kind, it follows that the first two terms, including the term containing the potential, vanish. Containing only the third term,

the equation now becomes the equation, which is conventionally called the Kolmogorov–Feller equation or the master equation. In this case, it is natural to introduce a potential in the form of some additional structure on an ultrametric space; this additional structure should be incorporated in the probability of transition per unit time, which in turn enters the master equation.

One way to introduce potential in the model of a p -adic random walk is to consider the master equation for the probability density of the form

$$\frac{df(x, t)}{dt} = \int_{\mathbb{Q}_p} \frac{u(y) f(y, t) - u(x) f(x, t)}{|x - y|_p^{\alpha+1}} d_p y, \quad (2)$$

where \mathbb{Q}_p is field of p -adic numbers, $d_p y$ is the Haar measure on \mathbb{Q}_p , α is positive parameter interpreted as the inverse temperature, $u(x)$ is positive definite function on \mathbb{Q}_p (potential function). We assume that the function $u(x)$ belongs to the class W_β , for some $\beta < \alpha$ i.e. satisfies the properties: 1) $|u(x)| \leq C(1 + |x|_p^\beta)$; 2) there exists n , such that $u(x + x') = u(x)$ for all $|x'|_p \leq p^{-n}$. The Cauchy problem for equation (2) is determined in the class of functions $f(x, t)$, which are continuously differentiable in t , and as a function of $x \in \mathbb{Q}_p$ belong to the class W_γ for $\gamma < \alpha - \beta$ uniformly with respect to t . In this work, equation (2) will be called the equation of p -adic random walk in a potential $u(x)$.

At present, methods of analytical solutions of equation (2) in potential of arbitrary form $u(x)$ are unknown. Nevertheless, it turns out that for a definite form of a power-law potential this equation has a precise analytic solution.

This article is devoted to the analytical solution of the Cauchy problem for equation (2) with the initial condition on the ring of integer p -adic numbers \mathbb{Z}_p for the specific choice of the potential function $u(x)$ depending only on p -adic norm $|x|_p$ of $x \in \mathbb{Q}_p$ as the power law of a special form. This model can be viewed as a generalization of the models used in previous works for description of protein conformational dynamics [13, 14, 15, 16, 17, 18, 19, 20, 21]. In Section 2 we show that, for such special potentials, the equation of p -adic random walk in a potential can be reduced to the equation of p -adic random walk with modified measure and reaction sources, whose support lies in \mathbb{Z}_p . The last equation will be the subject of research of the last two sections. In Section 3 we examine the equation of a p -adic random walk with arbitrary measure and a source in \mathbb{Z}_p . Using the basis of eigenfunctions of the Vladimirov operator with modified measure, as was constructed in our recent paper [23], we find an analytic solution of the Cauchy problem for such an equation with an initial condition in \mathbb{Z}_p . In Section 4 we particularize the solution of this equation in the case when the measure is induced by a potential of special form, which was considered in Section 2. In Section 5 we examine the asymptotic behaviour of the so-obtained solution for the distribution function for large times. Namely, we establish that the distribution function tends, as $t \rightarrow \infty$, to the stationary solution according to a law that is bounded from above and below by power laws with the same exponent. Our main conclusion is that the introduction of a potential in the ultrametric model of conformational dynamics of protein conserves the power-law behaviour of the relaxation curves for large times. In Appendixes A, B, C and D we state and prove a number of assertions that are used for solving the principal equation of the model and in obtaining an asymptotic estimate of the solution thus obtained for large times.

2 Particular case of a power-law potential

We consider the equation of p -adic random walk in the potential (2) when the function $u(x)$ is in the class W_β and has the form

$$u(x) = a\Omega(|x|_p) + b|x|_p^\beta(1 - \Omega(|x|_p)), \quad 0 \leq \beta < \alpha, \quad (3)$$

where

$$\Omega(|x|_p) = \begin{cases} 1, & |x|_p \leq 1, \\ 0, & |x|_p > 1. \end{cases}$$

We denote $L_{l.c.}^2(\mathbb{Q}_p, d_p x) \subset W_0$ the class of locally constant functions of functions square-integrable with respect to p -adic Haar measure $d_p x$. Also we denote $L_{l.c.}^2(\mathbb{Q}_p, u(x) d_p x)$ the class of locally constant functions of functions square-integrable with respect to measure $u(x) d_p x$. Let us state the Cauchy problem for (2) with the potential (3) and the initial condition

$$f(x, 0) = f_0(x) \quad (4)$$

in the class $L_{l.c.}^2(\mathbb{Q}_p, d_px)$. The following result holds.

Lemma 1. *Let constants a, b, α and β satisfy the conditions*

$$a = b \frac{1 - p^{-1}}{1 - p^{-\alpha}}, \quad (5)$$

$$\beta = \alpha - 1. \quad (6)$$

Then the solution of the Cauchy problem for equation

$$\frac{df(x, t)}{dt} = \int_{\mathbb{Q}_p} \frac{f(y, t) - f(x, t)}{|x - y|_p^{\alpha+1}} u(y) d_py + \lambda \Omega(|x|_p) f(x, t), \quad (7)$$

with the initial condition (4) (here $\lambda = b\Gamma_p(\alpha)\Gamma_p(-\alpha)$ and $\Gamma_p(-\alpha) = \frac{1 - p^{-\alpha-1}}{1 - p^{-\alpha}}$ is p -adic gamma function) in the class $L_{l.c.}^2(\mathbb{Q}_p, u(x) d_px)$ is the solution of the Cauchy problem for equation (2) in the class $L_{l.c.}^2(\mathbb{Q}_p, d_px)$ with the initial condition (4).

Proof. First of all, note that $L_{l.c.}^2(\mathbb{Q}_p, u(x) d_px) \subset L_{l.c.}^2(\mathbb{Q}_p, d_px)$. Next, we show that under conditions (5) and (6) the equation (2) can be transformed to the form (7). The equation (2) can be rewritten in the form of the equation of p -adic random walk with the modified measure $u(y) d_py$ and reaction source as follows

$$\frac{df(x, t)}{dt} = \int_{\mathbb{Q}_p} \frac{f(y, t) - f(x, t)}{|x - y|_p^{\alpha+1}} u(y) d_py + J(x) f(x, t). \quad (8)$$

where

$$J(x) = \Gamma_p(-\alpha) D^\alpha u(x), \quad (9)$$

D^α is the pseudodifferential Vladimirov operator [6]

$$D^\alpha \varphi(x) = \frac{1}{\Gamma_p(-\alpha)} \int_{\mathbb{Q}_p} \frac{\varphi(y) - \varphi(x)}{|x - y|_p^{\alpha+1}} d_py.$$

To calculate $D^\alpha u(x)$ we find the p -adic Fourier image $\tilde{u}(k)$ of the function $u(x)$. Using the relations

$$\int_{\mathbb{Q}_p} d_px \Omega(|x|_p) \chi(kx) = \Omega(|k|_p)$$

and

$$\begin{aligned} & \int_{\mathbb{Q}_p} d_px |x|_p^\beta (1 - \Omega(|x|_p)) \chi(kx) = \\ & = \Gamma_p(\alpha\beta + 1) |k|_p^{-\beta-1} - \frac{1 - p^{-1}}{1 - p^{-\alpha\beta-1}} \Omega(|k|_p) - \Gamma_p(\alpha\beta + 1) |k|_p^{-\beta-1} (1 - \Omega(|k|_p)) \end{aligned}$$

we find that

$$\begin{aligned} \tilde{u}(k) &= \int_{\mathbb{Q}_p} d_px u(x) \chi(kx) = \\ &= a\Omega(|k|_p) + b \left(-\frac{1 - p^{-1}}{1 - p^{-\beta-1}} \Omega(|k|_p) + \Gamma_p(\beta + 1) |k|_p^{-\beta-1} \Omega(|k|_p) \right). \end{aligned}$$

We have

$$D^\alpha u(x) = \int_{\mathbb{Q}_p} d_p k |k|_p^\alpha \tilde{u}(k) \chi(-kx),$$

and hence

$$D^\alpha u(x) = \left(a - b \frac{1 - p^{-1}}{1 - p^{-\beta-1}} \right) D^\alpha \Omega(|x|_p) + b \Gamma_p(\alpha) D^{-\beta+\alpha-1} \Omega(|x|_p)$$

In view of conditions (5) and (6), we find that

$$D^\alpha u(x) = b \Gamma_p(\alpha) \Omega(|x|_p).$$

As a result,

$$J(x) = \int_{\mathbb{Q}_p} \frac{u(y) - u(x)}{|x - y|_p^{\alpha+1}} dy = \Gamma_p(-\alpha) D^\alpha u(x) = b \Gamma_p(\alpha) \Gamma_p(-\alpha) \Omega(|x|_p) = \lambda \Omega(|x|_p)$$

and so (8) assumes the form (7), which completes the proof of Lemma 1. \square

Remark 1. Note that in the framework of assumptions (5) and (6) of Lemma 1 we have $\lambda > 0$ for $\alpha > 1$, $\lambda < 0$ for $\alpha < 1$, and $\lambda = 0$ for $\alpha = 1$. Moreover, for $\alpha < 1$ we have $\beta < 0$, hence we will study only the case $\alpha > 1$.

Remark 2. For $\alpha > 2$ the equation has the normalized to unity stationary solution

$$f^{(st)}(x) = \left(\frac{1 - p^{-\alpha}}{1 - p^{-1}} + \frac{1 - p^{-1}}{p^{\alpha-2} - 1} \right)^{-1} \left(\frac{1 - p^{-\alpha}}{1 - p^{-1}} \Omega(|x|_p) + |x|_p^{-\alpha+1} (1 - \Omega(|x|_p)) \right). \quad (10)$$

Later we will show that the equilibrium solution of equation (2)

$$f^{(eq)}(x) = \lim_{t \rightarrow \infty} f(x, t)$$

is

$$f^{(eq)}(x) = \begin{cases} f^{(st)}(x), & \alpha > 2, \\ 0, & 1 \leq \alpha \leq 2. \end{cases}$$

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Solution of the Cauchy problem for the equation (7) in the class $L_{l.c.}^2(\mathbb{Q}_p, u(x) d_p x)$ can be found analytically. Below we shall find and explore an analytical solution of this problem for the homogeneous initial condition in \mathbb{Z}_p .

3 The general solution of the equation of p -adic random walk with general modified measure $u(x) d_p x$ and reaction source in \mathbb{Z}_p

In this section we solve the Cauchy problem of equation

$$\frac{df(x, t)}{dt} = \int_{\mathbb{Q}_p} \frac{f(y, t) - f(x, t)}{|x - y|_p^{\alpha+1}} u(y) d_p y + \lambda \Omega(|x|_p) f(x, t) \quad (11)$$

with the initial condition

$$f(x, 0) = \Omega(|x|_p) \quad (12)$$

and an arbitrary function $u(x)$. In finding the solution we shall use a number of auxiliary results, which will be formulated and proved in Appendix A.

The solution of the Cauchy problem (11)–(12) will be searched in the subclass of locally constant functions of the space $L_{norm}^2(\mathbb{Q}_p, u(x) d_p x)$ (see Appendix A). Passing to the Laplace images, we obtain

$$s\tilde{f}(x, s) = \Omega(|x|_p) + D_{u(x)}^\alpha \tilde{f}(x, s) + \lambda \Omega(|x|_p) \tilde{f}(x, s). \quad (13)$$

Employing Theorem 1 from Appendix A, we expand the function $\tilde{f}(x, s)$ in basis (39),

$$\tilde{f}(x, s) = \sum_{i=-\infty}^{\infty} \tilde{f}_i(s) \phi_i(x). \quad (14)$$

An application of Theorems 2 and 3 and Corollary 1 from Appendix A shows that

$$\begin{aligned} s\tilde{f}_k(s) &= V_0 V_{k-1}^{-\frac{1}{2}} \left(1 - \frac{V_{k-1}}{V_k}\right)^{\frac{1}{2}} - \left(1 - p^{-(\alpha+1)}\right) \left(\sum_{i=k}^{\infty} p^{-i(\alpha+1)} V_i\right) \tilde{f}_k(s) + \\ &+ \lambda \left(V_0 V_{k-1}^{-1} \left(1 - \frac{V_{k-1}}{V_k}\right)^{-1} \left(1 - 3\frac{V_{k-1}}{V_k} + 2\frac{V_{k-1}^2}{V_k^2}\right) + \sum_{i=k}^{\infty} V_0 V_{i-1}^{-1} \left(1 - \frac{V_{i-1}}{V_i}\right)\right) \tilde{f}_k(s) + \\ &- 2\lambda V_0 V_{k-1}^{-1} \left(1 - \frac{V_{k-1}}{V_k}\right) \tilde{f}_k(s) + \\ &+ \lambda V_0 V_{k-1}^{-\frac{1}{2}} \left(1 - \frac{V_{k-1}}{V_k}\right)^{\frac{1}{2}} \sum_{i=1}^{\infty} \tilde{f}_i(s) V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}} \end{aligned} \quad (15)$$

for $k \geq 1$ and

$$s\tilde{f}_k(s) = -\left(1 - p^{-(\alpha+1)}\right) \left(\sum_{i=k}^{\infty} p^{-i(\alpha+1)} V_i\right) \tilde{f}_k(s) + \lambda \sum_{i=k+1}^{\infty} \sum_{i=k+1}^{\infty} V_0 V_{i-1}^{-1} \left(1 - \frac{V_{i-1}}{V_i}\right) \tilde{f}_k(s) \quad (16)$$

for $k < 1$.

Equation (15) can be rewritten as

$$\tilde{f}_k(s) = g_k(s) + \lambda g_k(s) \sum_{i=1}^{\infty} \tilde{f}_i(s) V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}} \quad (17)$$

where

$$\begin{aligned} g_k(s) &= V_0 V_{k-1}^{-\frac{1}{2}} \left(1 - \frac{V_{k-1}}{V_k}\right)^{\frac{1}{2}} \left[s + \left(1 - p^{-(\alpha+1)}\right) \left(\sum_{i=k}^{\infty} p^{-i(\alpha+1)} V_i\right) - \right. \\ &- \lambda V_0 V_{k-1}^{-1} \left(1 - \frac{V_{k-1}}{V_k}\right)^{-1} \left(1 - 3\frac{V_{k-1}}{V_k} + 2\frac{V_{k-1}^2}{V_k^2}\right) - \\ &- \left. \lambda \sum_{i=k}^{\infty} V_0 V_{i-1}^{-1} \left(1 - \frac{V_{i-1}}{V_i}\right) + 2\lambda V_0 V_{k-1}^{-1} \left(1 - \frac{V_{k-1}}{V_k}\right) \right]^{-1}. \end{aligned} \quad (18)$$

Setting

$$\begin{aligned} F(s) &= \sum_{i=1}^{\infty} \tilde{f}_i(s) V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}}, \\ G(s) &= \sum_{i=1}^{\infty} g_i(s) V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}}, \end{aligned}$$

we write equation (17) in the form

$$\tilde{f}_k(s) = g_k(s) (1 + \lambda F(s)). \quad (19)$$

Multiplying (19) by $V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}}$ and summing in k , we find that

$$F(s) = G(s) + \lambda G(s) F(s),$$

and hence,

$$F(s) = \frac{G(s)}{1 - \lambda G(s)}. \quad (20)$$

Substituting (20) into (19), this gives

$$\tilde{f}_k(s) = \frac{g_k(s)}{1 - \lambda G(s)}. \quad (21)$$

From equation (16) we find that

$$\tilde{f}_k(s) = 0, \quad k < 1. \quad (22)$$

In view of (21) and (22), function (14) assumes the form

$$\tilde{f}(x, s) = \sum_{i=1}^{\infty} \frac{g_i(s)}{1 - \lambda G(s)} \phi_i(x) \quad (23)$$

Expression (23) is the general solution of the equation of p -adic random walk with source in \mathbb{Z}_p with the initial distribution in Laplace images.

4 Solution in the case of a modified measure induced by a power-law potential

In this section we particularize the solution (23) with the following function $u(x)$:

$$u(x) = b \left(\frac{1 - p^{-1}}{1 - p^{-\alpha}} \Omega(|x|_p) + |x|_p^{\alpha-1} (1 - \Omega(|x|_p)) \right). \quad (24)$$

Calculating the volumes of balls (40) with respect to the measure $u(x) d_p x$, this gives

$$V_i = b (1 - p^{-1}) \frac{p^{\alpha(i+1)}}{p^{\alpha} - 1}. \quad (25)$$

Here, the basis functions (39) assume the form

$$\phi_i(x) = \left(b (1 - p^{-1}) p^{\alpha(i-1)} \right)^{-\frac{1}{2}} \left(\Omega(|x|_p p^{-i+1}) - p^{-\alpha} \Omega(|x|_p p^{-i}) \right). \quad (26)$$

Calculating functions (18), we obtain

$$g_k(s) = (bp^{\alpha} (1 - p^{-1}))^{\frac{1}{2}} \frac{p^{-\alpha k/2}}{s - bp^{-k} p^{\alpha} \Gamma_p(-\alpha)}. \quad (27)$$

The function $G(s)$ reads as

$$G(s) = \sum_{i=1}^{\infty} \frac{(p^{\alpha} - 1) p^{-\alpha i}}{s - bp^{-i} p^{\alpha} \Gamma_p(-\alpha)}. \quad (28)$$

Substituting (27) and (28) into (21), we find that

$$\tilde{f}_k(s) = (bp^{\alpha} (1 - p^{-1}))^{\frac{1}{2}} \frac{p^{-\alpha k/2}}{s - bp^{-k} p^{\alpha} \Gamma_p(-\alpha)} \frac{1}{1 - \lambda \sum_{i=1}^{\infty} \frac{(p^{\alpha} - 1) p^{-\alpha i}}{s - bp^{-i} p^{\alpha} \Gamma_p(-\alpha)}}. \quad (29)$$

Next, in view of (26), we have

$$\tilde{f}(x, s) = \sum_{k=1}^{\infty} \frac{p^{-\alpha k} p^{\alpha} (\Omega(|x|_p p^{-k+1}) - p^{-\alpha} \Omega(|x|_p p^{-k}))}{(s - bp^{-k} p^{\alpha} \Gamma_p(-\alpha)) \left(1 - \lambda \sum_{i=1}^{\infty} \frac{(p^{\alpha} - 1) p^{-\alpha i}}{s - bp^{-i} p^{\alpha} \Gamma_p(-\alpha)}\right)}.$$

Here, the Laplace image of the probability of finding the trajectory of a random process in the support of the initial distribution is as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} \tilde{f}(x, s) d_p x &= \sum_{k=1}^{\infty} \frac{p^{-\alpha k} (p^{\alpha} - 1)}{(s - bp^{-k} p^{\alpha} \Gamma_p(-\alpha)) \left(1 - \lambda \sum_{i=1}^{\infty} \frac{(p^{\alpha} - 1) p^{-\alpha i}}{s - bp^{-i} p^{\alpha} \Gamma_p(-\alpha)}\right)} = \\ &= \frac{G(s)}{1 - \lambda G(s)} = F(s). \end{aligned} \quad (30)$$

5 Asymptotic behaviour of the distribution function as $t \rightarrow \infty$

We first examine the asymptotic behaviour as $t \rightarrow \infty$ of the probability of finding the trajectory in the support of the initial distribution, which we denote by $S(t)$ and which is the Laplace preimage of function (30). To this aim we need to find the poles and residues of the function $G(s)$ (see (28)). Clearly, the function $G(s)$ is positive at zero, and besides,

$$G(0) = -\frac{(p^{\alpha} - 1)}{bp^{\alpha} \Gamma_p(-\alpha)} \sum_{i=1}^{\infty} p^{-(\alpha-1)i} = \frac{1}{\lambda}.$$

The function $G(s)$ vanishes at infinity

$$\lim_{s \rightarrow \infty} G(s) = 0.$$

The derivative of $G(s)$ is negative

$$\frac{d}{ds} G(s) = -\sum_{i=1}^{\infty} \frac{(p^{\alpha} - 1) p^{-\alpha i}}{(s - bp^{-i} p^{\alpha} \Gamma_p(-\alpha))^2} < 0,$$

and hence the function $G(s)$ is decreasing. The function $G(s)$ has simple poles at the points $s_k = bp^{-k} p^{\alpha} \Gamma_p(-\alpha) < 0$, which correspond to the zeros of the function $F(s)$. To the left from the poles $s_k = bp^{-k} p^{\alpha} \Gamma_p(-\alpha) < 0$ the function $G(s)$ is negative, and to the right, is positive. The points of intersection of the function $G(s)$ with the line $G = \frac{1}{\lambda}$ correspond to the poles of the function $F(s)$. Besides, the function $F(s)$ has an evident pole at the point $s = 0$, at which the function $G(s)$ assumes the value $\frac{1}{\lambda}$. Clearly, all the poles of the function $F(s)$ are negative, except for the pole $s = 0$. The function $G(s)$ vanishes at the infinity, and hence so is the function $F(s)$:

$$\lim_{s \rightarrow \infty} F(s) = 0.$$

Let $s = -\lambda_i$, $i \in \mathbb{Z}_+$, be the poles of the function $F(s)$. Note that the function $F(s)$ is not meromorphic (the point $s = 0$ is a point of condensation of simple poles). Nevertheless, one may show (see [18] for details) that the function $F(s)$ can be expanded in simple poles,

$$F(s) = \frac{b_0}{s} + \sum_{k=1}^{\infty} \frac{b_k}{s + \lambda_k}.$$

Clearly, λ_i satisfy the condition

$$-bp^{-(i+1)} p^{\alpha} \Gamma_p(-\alpha) \leq \lambda_i \leq -bp^{-i} p^{\alpha} \Gamma_p(-\alpha).$$

We write λ_i as

$$\lambda_i = -bp^{-i-1} p^{\alpha} \Gamma_p(-\alpha) (1 + \delta_i),$$

where δ_i is a small positive error satisfying the condition

$$0 < \delta_i < p - 1. \quad (31)$$

From the equation for the poles,

$$G(-\lambda_k) = \frac{1}{\lambda},$$

it follows that

$$\sum_{i=1}^{\infty} \frac{p^{-\alpha i}}{p^{-k-1} + p^{-k-1}\delta_k - p^{-i}} = -\frac{1}{p^{\alpha-1} - 1}. \quad (32)$$

Equation (32) enables one to obtain an asymptotic estimate for δ_k as $k \rightarrow \infty$ (see Appendix B, formula (53)).

The residues at the poles $s = -\lambda_i$ of the function $F(s)$ are as follows:

$$\begin{aligned} b_k &= \lim_{s \rightarrow -\lambda_k} (s + \lambda_k) F(s) = -\frac{\lambda^{-2}}{G'(-\lambda_k)} = \\ &= \frac{p^\alpha - 1}{(p^{\alpha-1} - 1)^2} \left[\sum_{i=1}^{\infty} \frac{p^{-\alpha i}}{(p^{-k-1} + p^{-k-1}\delta_k - p^{-i})^2} \right]^{-1}. \end{aligned} \quad (33)$$

Equation (33) in view of asymptotic estimate for δ_k enables one to obtain an asymptotic estimate for b_k as $k \rightarrow \infty$ (see Appendix C, formula (55)). This estimate can be written as

$$\begin{aligned} C_{\min}(\alpha) p^{-|\alpha-2|k} (1 + \delta_{\alpha,2} (k^{-2} - 1)) (1 + o(1)) < \\ < b_k < C_{\max}(\alpha) p^{-|\alpha-2|k} (1 + \delta_{\alpha,2} (k^{-2} - 1)) (1 + o(1)), \end{aligned} \quad (34)$$

where $o(1)$ is an infinitesimal quantity as $k \rightarrow \infty$ and

$$\begin{aligned} C_{\min}(\alpha) &= \begin{cases} \frac{(p^\alpha - 1)(1 - p^{-1})^2(1 - p^{-\alpha+2})^2}{p^{2-\alpha}(p^{\alpha-1} - 1)^2}, & \alpha > 2, \\ \frac{(p^2 - 1)}{(p - 1)^2}, & \alpha = 2, \\ \frac{(p^\alpha - 1)\delta_{\max}^2}{p^{2-\alpha}(p^{\alpha-1} - 1)^2}, & 1 < \alpha < 2, \end{cases} \\ C_{\max}(\alpha) &= (1 + o(1)) \begin{cases} \frac{(p^\alpha - 1)(1 - p^{-\alpha+2})^2}{p^{2-\alpha}(p^{\alpha-1} - 1)^2}, & \alpha > 2, \\ \frac{(1 - p^{-2})(1 - p^{-1})^2}{(p - 1)^2}, & \alpha = 2, \\ \frac{(p^\alpha - 1)\delta_{\min}^2}{p^{2-\alpha}(p^{\alpha-1} - 1)^2}, & 1 < \alpha < 2. \end{cases} \end{aligned}$$

The function $S(t)$ is as follows:

$$S(t) = b_0 + \sum_{k=1}^{\infty} b_k \exp(-\lambda_k t). \quad (35)$$

Using estimates for b_k and for λ_k one can estimate $S(t)$ as

$$\begin{aligned} b_0 + C_{\min}(\alpha) \sum_k p^{-|\alpha-2|k} (1 + \delta_{\alpha,2} (k^{-2} - 1)) \exp(-wp^{-k}p^{-1}t) (1 + o(1)) < \\ < S(t) < b_0 + C_{\max}(\alpha) \sum_k p^{-|\alpha-2|k} (1 + \delta_{\alpha,2} (k^{-2} - 1)) \exp(-wp^{-k}p^{-1}t) (1 + o(1)), \end{aligned}$$

where

$$w = -bp^\alpha \Gamma_p(-\alpha).$$

An application of Lemma 4 from Appendix D gives, as $t \rightarrow \infty$,

$$\begin{aligned} & b_0 + A(\alpha) t^{-|\alpha-2|} \left(1 + \delta_{\alpha,2} \left(\left(\frac{\ln p}{\ln t} \right)^2 - 1 \right) \right) (1 + o(1)) < \\ & < S(t) < b_0 + B(\alpha) t^{-|\alpha-2|} \left(1 + \delta_{\alpha,2} \left(\left(\frac{\ln p}{\ln t} \right)^2 - 1 \right) \right) (1 + o(1)) \end{aligned} \quad (36)$$

where $A(\alpha)$ and $B(\alpha)$ are defined by

$$\begin{aligned} A(\alpha) &= C_{\min}(\alpha) p^{-|\alpha-2|} (\ln p)^{-1} (wp^{-1})^{-|\alpha-2|} \Gamma(|\alpha-2|), \\ B(\alpha) &= C_{\max}(\alpha) p^{-|\alpha-2|} (\ln p)^{-1} (wp^{-1})^{-|\alpha-2|} \Gamma(|\alpha-2|) \end{aligned}$$

and $o(1)$ is an infinitesimal quantity as $t \rightarrow \infty$.

We note that

$$b_0 = \begin{cases} \frac{(p^\alpha - 1)(p^{\alpha-2} - 1)}{(p^{\alpha-1} - 1)^2} & \alpha > 2, \\ 0, & \alpha \leq 2, \end{cases}$$

is nothing else but the equilibrium solution at the point $x = 0$, i.e.

$$\lim_{t \rightarrow \infty} f(0, t) = \lim_{t \rightarrow \infty} S(t) = b_0,$$

which for $\alpha > 2$ concides with the normalized stationary solution $f^{(st)}(x)$ at the point $x = 0$

$$\lim_{t \rightarrow \infty} S(t) = b_0.$$

Let us discuss the asymptotic behaviour of the distribution function $f(x, t)$ as $t \rightarrow \infty$ in the domain $|x|_p > 1$. For the Laplace image of $f(x, t)$ we have

$$\tilde{f}(x, s) = \frac{K(s, |x|_p) + G(s)}{1 - \lambda G(s)}, \quad (37)$$

where the function $K(s, |x|_p)$ is defined as

$$K(s, |x|_p) = -\frac{p^\alpha |x|_p^{-\alpha}}{(s - b|x|_p^{-1} p^\alpha \Gamma_p(-\alpha))} - \sum_{i=1}^{\ln |x|_p / \ln p} \frac{p^{-\alpha i} (p^\alpha - 1)}{(s - b p^{-i} p^\alpha \Gamma_p(-\alpha))},$$

and the function $G(s)$ is defined by formula (28). Clearly, the function (37) has a pole at the point $s = 0$, at which the residue $b_0(x)$ is the equilibrium solution of equation (7), which for $\alpha > 2$ concides with the normalized stationary solution $f^{(st)}(x)$. From (37) it follows that the function $\tilde{f}(x, s)$ also has poles λ_k , which coincide with the poles of the function $F(s)$. The residues $b_k(x)$ at these poles are as follows:

$$b_k(|x|_p) = \lim_{s \rightarrow -\lambda_k} (s + \lambda_k) \frac{K(s, |x|_p) + G(s)}{(1 - \lambda G(s))}. \quad (38)$$

Formula (38) can be rewritten as

$$b_k(|x|_p) = (h_k(|x|_p) + 1) b_k,$$

where

$$h_k(|x|_p) = (p^\alpha - 1) \left(\frac{p^\alpha |x|_p^{-\alpha}}{(p^{-k-1} + p^{-k-1} \delta_k - |x|_p^{-1})} + \sum_{i=1}^{\ln |x|_p / \ln p} \frac{p^{-\alpha i} (p^\alpha - 1)}{(p^{-k-1} + p^{-k-1} \delta_k - p^{-i})} \right)$$

and b_k is the residue at the pole $s = -\lambda_k$ of the function $F(s)$. For any fixed $|x|_p$, the function $h_k(|x|_p)$ has finite nonzero limit as $k \rightarrow \infty$. Hence, as $k \rightarrow \infty$ the behaviour of the residues $b_k(|x|_p)$ and b_k are equal up to a factor depending on $|x|_p$. For this reason, if $t \rightarrow \infty$ and for any $x \in \mathbb{Q}_p$ the function $f(x, t)$ tends to the equilibrium solution in accordance with the same law as the function $S(t)$.

6 Conclusion

In the present paper we consider the equation of p -adic random walk in a potential, which has the form (2). It is shown that in the case when the function of potential has a power form (3) and in the particular case when the parameters are related by (5) and (6), the equation (2) can be solved analytically. We found the solution to the Cauchy problem of this equation with the initial condition in \mathbb{Z}_p and examined its asymptotic behavior at large times. Moreover, we prove that, as $t \rightarrow \infty$, the distribution function tends to the stationary solution according to a law bounded from above and below by power laws with the same exponent. The analysis of numerical solutions of the p -adic random walk in a potential [24], which we not present here, shows that, for other values of the parameter β near the value $\beta = \alpha - 1$, the power-law behaviour of the distribution function is conserved for large times.

The results obtained in this work provide a basis for the belief that the introduction of a power-law potential in the p -adic model of the conformational dynamics of protein conserves the power-law behaviour of the relaxation curves for large times. This fact may serve as a justification of the conclusion that in a model that is capable of describing experiments in spectral diffusion in proteins and experiments in binding kinetics of CO by myoglobin (in which the relaxation has power-law behaviour), the introduction of a power-law potential may not be necessary. We admit, however, that in more precise experiments on the conformational dynamics of protein the potential may somehow exhibit itself. It is quite possible that the potential may play a certain role in models of prototypes of molecular nanomachines that are based on processes of p -adic random walk. One of such models, which was considered in [19], was based on a process of p -adic random walk without potential, but on the compact set $B_r \subset \mathbb{Q}_p$. Such a model can be physically interpreted as a model on \mathbb{Q}_p in a potential $u(x)$ of a special form, which is constant for $x \in B_r$ and is infinite for $x \notin B_r$.

Acknowledgements

The author is deeply indebted to Prof. I. V. Volovich (Steklov Mathematical Institute, Russian Academy of Sciences) for their careful reading of the manuscript, discussion of the results, and a number of useful comments. The author is also grateful to Prof. A. R. Alimov (Faculty of Mechanics and Mathematics, Moscow State University) for his assistance in preparing the manuscript and a number of helpful comments.

The work was partially supported by the Ministry of Education and Science of the Russian Federation under the Competitiveness Enhancement Program of SSAU for 2013–2020.

Appendix A. Some auxiliary results

Let $u(x)$ is positive $d_p x$ -locally integrable function on \mathbb{Q}_p . We denote by $L_{norm}^2(\mathbb{Q}_p, u(x) d_p x)$ the class of functions on \mathbb{Q}_p of the form $f(x) = f(|x|_p)$ with finite integral

$$\int_{\mathbb{Q}_p} d_p y u(y) |f(y)|^2.$$

Theorem 1. *The family of functions*

$$\phi_i(x) = \left(V_{i-1} \left(1 - \frac{V_{i-1}}{V_i} \right) \right)^{-\frac{1}{2}} \left(\Omega(|x|_p p^{-i+1}) - \frac{V_{i-1}}{V_i} \Omega(|x|_p p^{-i}) \right), \quad (39)$$

where $i \in \mathbb{Z}$ and

$$V_i = \int_{\mathbb{Q}_p} m(y) d_p y \Omega(|y|_p p^{-i}), \quad (40)$$

forms an orthonormal basis for $L_{norm}^2(\mathbb{Q}_p, u(x) d_p x)$.

Proof. That the system of functions (39) is orthonormal

$$\int_{\mathbb{Q}_p} \phi_i(x) \phi_j(x) m(x) d_p x = \delta_{ij}$$

is verified directly. Let us show that (39) is a basis for $L_{norm}^2(\mathbb{Q}_p, u(x) d_p x)$. Clearly, the countable family of the supports of all p -adic spheres

$$S_r = \{x \in \mathbb{Q}_p : |x|_p = p^r\}$$

of the form

$$\delta(|x|_p - p^r) = \begin{cases} 1, & x \in S_r, \\ 0, & x \notin S_r \end{cases}$$

forms a basis for $L_{norm}^2(\mathbb{Q}_p, u(x) d_p x)$. We have $\delta(|x|_p - p^r) = \Omega(|x|_p p^{-r}) - \Omega(|x|_p p^{-r+1})$, and so it suffices to show that, for any r , the function $\Omega(|x|_p p^{-r})$ can be expanded in functions (39). We claim that the norm of the following function is zero. Indeed,

$$\begin{aligned} & \left\| \Omega(|x|_p p^{-r}) - V_r \sum_{i=r+1}^{\infty} V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}} \phi_i(x) \right\|^2 = \\ &= \int_{\mathbb{Q}_p} \Omega(|x|_p p^{-r}) m(x) d_p x - \\ & - 2V_r \sum_{i=r+1}^{\infty} V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}} \int_{\mathbb{Q}_p} \Omega(|x|_p p^{-r}) \phi_i(x) m(x) d_p x + \\ & + V_r^2 \sum_{i=r+1}^{\infty} V_{i-1}^{-1} \left(1 - \frac{V_{i-1}}{V_i}\right) \int_{\mathbb{Q}_p} (\phi_i(x))^2 m(x) d_p x = \\ &= V_r - 2V_r \sum_{i=r+1}^{\infty} V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}} V_{i-1}^{-\frac{1}{2}} V_r \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}} + \\ & + V_r^2 \sum_{i=r+2}^{\infty} V_{i-1}^{-1} \left(1 - \frac{V_{i-1}}{V_i}\right) = \\ &= V_r - V_r^2 \left(\sum_{i=r+1}^{\infty} V_{i-1}^{-1} - \sum_{i=r+2}^{\infty} V_i^{-1} \right) = 0 \end{aligned}$$

As a result, we have

$$\Omega(|x|_p p^{-r}) = V_r \sum_{i=r+1}^{\infty} V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}} \phi_i(x), \quad (41)$$

completing the proof of the theorem. \square

Any function $f(x)$ from $L_{norm}^2(\mathbb{Q}_p, u(x) d_p x)$ can be expanded as

$$f(x) = \sum_{i=-\infty}^{\infty} f_i \phi_i(x),$$

where

$$f_i = \int_{\mathbb{Q}_p} \phi_i(x) f(x) m(x) d_p x.$$

We note that functions (39) have the property

$$\int_{\mathbb{Q}_p} \phi_i(x) m(x) d_p x = 0.$$

The following result holds [23].

Theorem 2. Functions (39) are eigenfunctions of the operator that generalizes the Vladimirov operator in $L_a^2(\mathbb{Q}_p, m(x) d_p x)$

$$D_{m(x)}^\alpha f(x) = \int_{\mathbb{Q}_p} d_p y m(y) \frac{f(y) - f(x)}{|x - y|_p^{\alpha+1}} \quad (42)$$

with the eigenvalues

$$\lambda_i = - \left(1 - p^{-(\alpha+1)}\right) \sum_{j=i}^{\infty} p^{-j(\alpha+1)} V_j.. \quad (43)$$

Lemma 2. The following formula holds

$$\begin{aligned} \int_{\mathbb{Q}_p} \phi_i(x) \phi_j(x) \phi_k(x) m(x) d_p x = \\ = \delta_{ij} \delta_{jk} V_{k-1}^{-\frac{1}{2}} \left(1 - \frac{V_{k-1}}{V_k}\right)^{-\frac{3}{2}} \left(1 - 3 \frac{V_{k-1}}{V_k} + 2 \frac{V_{k-1}^2}{V_k^2}\right) + \\ + \delta_{ij} \delta_{i < k} V_{k-1}^{-\frac{1}{2}} \left(1 - \frac{V_{k-1}}{V_k}\right)^{\frac{1}{2}} + \delta_{ik} \delta_{i < j} V_{j-1}^{-\frac{1}{2}} \left(1 - \frac{V_{j-1}}{V_j}\right)^{\frac{1}{2}} + \\ + \delta_{jk} \delta_{j < i} V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}}, \end{aligned}$$

where

$$\delta_{j < i} = \begin{cases} 1, & j < i, \\ 0, & j \geq i. \end{cases}$$

The lemma is proved by direct calculation.

Theorem 3. Let $f(x), g(x) \in L_{norm}^2(\mathbb{Q}_p, u(x) d_p x)$,

$$f(x) = \sum_{i=-\infty}^{\infty} f_i \phi_i(x), \quad g(x) = \sum_{i=-\infty}^{\infty} g_i \phi_i(x).$$

Then

$$f(x) g(x) = \sum_{k=-\infty}^{\infty} h_k \phi_k(x),$$

where

$$\begin{aligned} h_k = f_k g_k V_{k-1}^{-\frac{1}{2}} \left(1 - \frac{V_{k-1}}{V_k}\right)^{-\frac{3}{2}} \left(1 - 3 \frac{V_{k-1}}{V_k} + 2 \frac{V_{k-1}^2}{V_k^2}\right) + \\ + V_{k-1}^{-\frac{1}{2}} \left(1 - \frac{V_{k-1}}{V_k}\right)^{\frac{1}{2}} \sum_{i=-\infty}^{k-1} f_i g_i + f_k \sum_{i=k+1}^{\infty} g_i V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}} + \\ + g_k \sum_{i=k+1}^{\infty} f_i V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}}. \end{aligned}$$

Proof. We write

$$f(x)g(x) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f_i g_j \phi_i(x) \phi_j(x) = \sum_{k=-\infty}^{\infty} h_k \phi_k(x),$$

where

$$h_k = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f_i g_j \int_{\mathbb{Q}_p} \phi_i(x) \phi_j(x) \phi_k(x) m(x) d_p x.$$

Now the theorem follows from Lemma 2. □

The following result is a direct corollary to Theorem 3 taking into account the expansion formula

$$\Omega(|x|_p) = \sum_{i=0}^{\infty} \frac{V_0}{V_i} \phi_k(x).$$

Corollary 1. *Let $f(x) \in L_{norm}^2(\mathbb{Q}_p, u(x) d_p x)$ and $f(x) = \sum_{k=-\infty}^{\infty} f_k \phi_k(x)$. Then the production of functions $\Omega(|x|_p) f(x)$ is expanded in the basis (39) as*

$$\Omega(|x|_p) f(x) = \sum_{k=-\infty}^{\infty} C_k \phi_k(x),$$

where

$$\begin{aligned} C_k &= f_k V_0 V_{k-1}^{-1} \left(1 - \frac{V_{k-1}}{V_k}\right)^{-1} \left(1 - 3 \frac{V_{k-1}}{V_k} + 2 \frac{V_{k-1}^2}{V_k^2}\right) + \\ &+ f_k \sum_{i=k}^{\infty} V_0 V_{i-1}^{-1} \left(1 - \frac{V_{i-1}}{V_i}\right) - 2 f_k V_0 V_{k-1}^{-1} \left(1 - \frac{V_{k-1}}{V_k}\right) + \\ &+ V_0 V_{k-1}^{-\frac{1}{2}} \left(1 - \frac{V_{k-1}}{V_k}\right)^{\frac{1}{2}} \sum_{i=1}^{\infty} f_i V_{i-1}^{-\frac{1}{2}} \left(1 - \frac{V_{i-1}}{V_i}\right)^{\frac{1}{2}} \end{aligned} \quad (44)$$

for $k \geq 1$ and

$$C_k = f_k \sum_{i=k+1}^{\infty} V_0 V_{i-1}^{-1} \left(1 - \frac{V_{i-1}}{V_i}\right) \quad (45)$$

for $k < 1$.

Appendix B. Estimate of δ_k for large k

To estimate δ_k for large k it is convenient to write equation (32) as

$$\frac{p^{-(\alpha-1)k} p^{-\alpha+1}}{\delta_k} = \sum_{i=1}^k \frac{p^{-(\alpha-1)i}}{1 - p^{-k-1+i} (1 + \delta_k)} - \sum_{i=k+2}^{\infty} \frac{p^{-\alpha i} p^{k+1}}{1 - p^{-i+k+1} (1 + \delta_k)} - \frac{1}{p^{\alpha-1} - 1}. \quad (46)$$

In what follows we shall require the following result.

Lemma 3. *If $0 < x < 1$, then*

$$\frac{1}{1-x} > 1+x. \quad (47)$$

If $0 < x < y < 1$, then

$$\frac{1}{1-x} \leq 1+cx, \quad (48)$$

where $c = \frac{1}{1-y}$.

Proof.

$$\frac{1}{1-x} - (1+x) = \frac{x^2}{1-x} > 0.$$

$$\frac{1}{1-x} - (1+cx) = \frac{(1-c)x + cx^2}{1-x} \leq x \frac{(1-c) + cy}{1-x} = x \frac{1-c(1-y)}{1-x} = \frac{1-1}{1-x} = 0.$$

□

Consider the case $\alpha > 2$. By (47) and (48) and Lemma 3 with $\alpha \neq 2$ we estimate the terms on the right of (46) as follows:

$$\begin{aligned} \sum_{i=1}^k \frac{p^{-(\alpha-1)i}}{1-p^{-k-1+i}(1+\delta_k)} &> \sum_{i=1}^k p^{-(\alpha-1)i} (1+p^{-k-1+i}(1+\delta_k)) = \\ &= \frac{1-p^{-(\alpha-1)k}}{p^{\alpha-1}-1} + (1+\delta_k) \frac{1-p^{-(\alpha-2)k}}{p^{\alpha-2}-1} p^{-k-1}, \\ \sum_{i=1}^k \frac{p^{-(\alpha-1)i}}{1-p^{-k-1+i}(1+\delta_k)} &\leq \sum_{i=1}^k p^{-(\alpha-1)i} \left(1 + \frac{1+\delta_k}{1-p^{-1}} p^{-k-1+i}\right) = \\ &= \frac{1-p^{-(\alpha-1)k}}{p^{\alpha-1}-1} + \frac{1+\delta_k}{p-1} \frac{1-p^{-(\alpha-2)k}}{p^{\alpha-2}-1} p^{-k}, \end{aligned}$$

Moreover, the following estimate holds:

$$\begin{aligned} - \sum_{i=k+2}^{\infty} \frac{p^{-\alpha i} p^{k+1}}{1-p^{-i+k+1}(1+\delta_k)} &> - \sum_{i=k+2}^{\infty} \frac{p^{-\alpha i} p^{k+1}}{1-p^{-1}(1+\delta_k)} = - \frac{p^{-(\alpha-1)k} p^{-2\alpha+1}}{(1-p^{-1})(1-p^{-\alpha})(1+\delta_k)}, \\ - \sum_{i=k+2}^{\infty} \frac{p^{-\alpha i} p^{k+1}}{1-p^{-i+k+1}(1+\delta_k)} &< - \sum_{i=k+2}^{\infty} p^{-\alpha i} p^{k+1} = - \frac{p^{-(\alpha-1)k} p^{-2\alpha+1}}{1-p^{-\alpha}} \end{aligned}$$

Applying these estimates, we have

$$\frac{p^{-(\alpha-1)k} p^{-\alpha+1}}{\delta_k} > - \frac{p^{-(\alpha-1)(k+1)}}{1-p^{-\alpha+1}} + \frac{(1+\delta_k)(1-p^{-(\alpha-2)k})}{p^{\alpha-2}-1} p^{-k-1} - \frac{p^{-(\alpha-1)k} p^{-2\alpha+1}}{(1-p^{-1})(1-p^{-\alpha})(1+\delta_k)}, \quad (49)$$

$$\frac{p^{-(\alpha-1)k} p^{-\alpha+1}}{\delta_k} < - \frac{p^{-(\alpha-1)k}}{p^{\alpha-1}-1} + \frac{1+\delta_k}{p-1} \frac{1-p^{-(\alpha-2)k}}{p^{\alpha-2}-1} p^{-k} - \frac{p^{-(\alpha-1)k} p^{-2\alpha+1}}{1-p^{-\alpha}}. \quad (50)$$

Next we need the following assumption on the behaviour of δ_k as $k \rightarrow \infty$. We make the ansatz that $\delta_k \rightarrow 0$ as $k \rightarrow \infty$ and solve these inequalities with this ansatz. Then, in the asymptotic limit $k \rightarrow \infty$, the last two inequalities may be written as

$$(1-p^{-1})(1-p^{-\alpha+2})p^{-(\alpha-2)k}(1+o(1)) < \delta_k < (1-p^{-\alpha+2})p^{-(\alpha-2)k}(1+o(1)), \quad \alpha > 2. \quad (51)$$

Consider the case $\alpha = 2$. In this case we estimate the terms on the right of (46) as

$$\begin{aligned} \sum_{i=1}^k \frac{p^{-(\alpha-1)i}}{1-p^{-k-1+i}(1+\delta_k)} &> \sum_{i=1}^k p^{-i} (1+p^{-k-1+i}(1+\delta_k)) = \\ &= \frac{1-p^{-k}}{p^{\alpha-1}-1} + (1+\delta_k) k p^{-k-1}, \\ \sum_{i=1}^k \frac{p^{-(\alpha-1)i}}{1-p^{-k-1+i}(1+\delta_k)} &\leq \sum_{i=1}^k p^{-i} \left(1 + \frac{1+\delta_k}{1-p^{-1}} p^{-k-1+i}\right) = \\ &= \frac{1-p^{-k}}{p^{\alpha-1}-1} + \frac{1+\delta_k}{p-1} k p^{-k}, \end{aligned}$$

$$\begin{aligned}
- \sum_{i=k+2}^{\infty} \frac{p^{-\alpha i} p^{k+1}}{1 - p^{-i+k+1} (1 + \delta_k)} &> - \sum_{i=k+2}^{\infty} \frac{p^{-2i} p^{k+1}}{1 - p^{-1} (1 + \delta_k)} = - \frac{p^{-k} p^{-3}}{(1 - p^{-1}) (1 - p^{-2}) (1 + \delta_k)}, \\
- \sum_{i=k+2}^{\infty} \frac{p^{-\alpha i} p^{k+1}}{1 - p^{-i+k+1} (1 + \delta_k)} &< - \sum_{i=k+2}^{\infty} p^{-2i} p^{k+1} = - \frac{p^{-k} p^{-3}}{1 - p^{-2}}.
\end{aligned}$$

Using these estimates, we may write

$$\begin{aligned}
\frac{p^{-k} p^{-1}}{\delta_k} &> - \frac{p^{-k-1}}{1 - p^{-1}} + (1 + \delta_k) k p^{-k-1} - \frac{p^{-k}}{(p-1)(p^2-1)(1+\delta_k)}, \\
\frac{p^{-k} p^{-1}}{\delta_k} &< - \frac{p^{-k}}{p^{\alpha-1} - 1} + \frac{1 + \delta_k}{p-1} k p^{-k} - \frac{p^{-k} p^{-3}}{1 - p^{-2}}.
\end{aligned}$$

We again make the ansatz that $\delta_k \rightarrow 0$ as $k \rightarrow \infty$. With this ansatz the last two inequalities read as

$$(1 - p^{-1}) k^{-1} (1 + o(1)) < \delta_k < k^{-1} (1 + o(1)), \quad \alpha > 2. \quad (52)$$

Consider the case $1 < \alpha < 2$. One may show that inequalities (49) and (50) with the ansatz $\delta_k \rightarrow 0$ as $k \rightarrow \infty$ lead to a contradiction. Hence, in this case we make a different ansatz that $\delta_k \rightarrow \text{const}$ as $k \rightarrow \infty$. This gives us the inequalities

$$\begin{aligned}
\frac{p^{-(\alpha-1)k} p^{-\alpha+1}}{\delta_k} &> - \frac{p^{-(\alpha-1)(k+1)}}{1 - p^{-\alpha+1}} + \frac{(1 + \delta_k) (1 - p^{-(\alpha-2)k})}{p^{\alpha-2} - 1} p^{-k-1} - \frac{p^{-(\alpha-1)k} p^{-2\alpha+1}}{(1 - p^{-1}) (1 - p^{-\alpha}) (1 + \delta_k)}, \\
\frac{p^{-(\alpha-1)k} p^{-\alpha+1}}{\delta_k} &< - \frac{p^{-(\alpha-1)k}}{p^{\alpha-1} - 1} + \frac{1 + \delta_k}{1 - p^{-1}} \frac{1 - p^{-(\alpha-2)k}}{p^{\alpha-2} - 1} p^{-k-1} - \frac{p^{-(\alpha-1)k} p^{-2\alpha+1}}{1 - p^{-\alpha}},
\end{aligned}$$

which may be written as

$$\begin{aligned}
\delta_k^2 + \left(1 - \frac{p^{-\alpha+2} - 1}{1 - p^{-\alpha+1}} - \frac{p^{-\alpha+2} - 1}{(1 - p^{-1})(p^\alpha - 1)} \right) \delta_k - (p^{-\alpha+2} - 1) &< 0, \\
\delta_k^2 + \left(1 - \frac{(1 - p^{-1})(p^{-\alpha+2} - 1)}{1 - p^{-\alpha+1}} - \frac{(1 - p^{-1})(p^{-\alpha+2} - 1)}{p^\alpha - 1} \right) \delta_k - (1 - p^{-1})(p^{-\alpha+2} - 1) &> 0.
\end{aligned}$$

We denote by δ_{\max} the maximal positive root of the equation

$$\delta_k^2 + \left(1 - \frac{p^{-\alpha+2} - 1}{1 - p^{-\alpha+1}} - \frac{p^{-\alpha+2} - 1}{(1 - p^{-1})(p^\alpha - 1)} \right) \delta_k - (p^{-\alpha+2} - 1) = 0$$

and denote by δ_{\min} the maximal positive root of the equation

$$\delta_k^2 + \left(1 - \frac{(1 - p^{-1})(p^{-\alpha+2} - 1)}{1 - p^{-\alpha+1}} - \frac{(1 - p^{-1})(p^{-\alpha+2} - 1)}{p^\alpha - 1} \right) \delta_k - (1 - p^{-1})(p^{-\alpha+2} - 1) = 0.$$

Then

$$\delta_{\min} (1 + o(1)) < \delta_k < \delta_{\max} (1 + o(1)).$$

One may show that $\lim_{\alpha \rightarrow 2} \delta_{\max} = 0$, $\lim_{\alpha \rightarrow 1} \delta_{\max} = \infty$ and $\lim_{\alpha \rightarrow 2} \delta_{\min} = 0$, $\lim_{\alpha \rightarrow 1} \delta_{\min} = \infty$.

Combining these results, we write down the lower and upper estimates for δ_k as $k \rightarrow \infty$ for different α as

$$D_{\min} < \delta_k < D_{\max}, \quad (53)$$

where

$$\begin{aligned}
D_{\min} &= (1 + o(1)) \begin{cases} (1 - p^{-1}) (1 - p^{-\alpha+2}) p^{-(\alpha-2)k}, & \alpha > 2, \\ (1 - p^{-1}) k^{-1}, & \alpha = 2, \\ \delta_{\min}, & 1 < \alpha < 2, \end{cases} \\
D_{\max} &= (1 + o(1)) \begin{cases} (1 - p^{-\alpha+2}) p^{-(\alpha-2)k}, & \alpha > 2, \\ k^{-1}, & \alpha = 2, \\ \delta_{\max}, & 1 < \alpha < 2. \end{cases}
\end{aligned}$$

Appendix C. Estimate of b_k for large k

Let us estimate the residues b_k as $k \rightarrow \infty$. To this aim we write (33) as

$$b_k = \frac{p^\alpha - 1}{(p^{\alpha-1} - 1)^2} \left[\sum_{i=1}^k \frac{p^{-\alpha i}}{(p^{-k-1} + p^{-k-1}\delta_k - p^{-i})^2} + \sum_{i=k+2}^{\infty} \frac{p^{-\alpha i}}{(p^{-k-1} + p^{-k-1}\delta_k - p^{-i})^2} + \frac{p^{2-\alpha}p^{-(\alpha-2)k}}{\delta_k^2} \right]^{-1}. \quad (54)$$

Taking into account that in the case $\alpha \geq 2$ we have $\delta_k \rightarrow 0$ as $k \rightarrow \infty$, we write

$$b_k = \frac{p^\alpha - 1}{(p^{\alpha-1} - 1)^2} \left[\sum_{i=1}^k \frac{p^{-\alpha i}}{(p^{-k-1} - p^{-i})^2} + \sum_{i=k+2}^{\infty} \frac{p^{-\alpha i}}{(p^{-k-1} - p^{-i})^2} + \frac{p^{2-\alpha}p^{-(\alpha-2)k}}{\delta_k^2} \right]^{-1} (1 + o(1)).$$

Let us estimate b_k from the above. First, we consider the case $\alpha > 2$. Using (53), we find that

$$b_k < \frac{p^\alpha - 1}{(p^{\alpha-1} - 1)^2} \left[\sum_{i=1}^k \frac{p^{-\alpha i}}{(p^{-k-1} - p^{-1})^2} + \sum_{i=k+2}^{\infty} \frac{p^{-\alpha i}}{(p^{-k-1})^2} + \frac{p^{2-\alpha}p^{-(\alpha-2)k}}{(1 - p^{-\alpha+2})^2} \right]^{-1} (1 + o(1)).$$

Summing and neglecting the terms that vanish for $k \rightarrow \infty$, this gives

$$b_k < \frac{(p^\alpha - 1)(1 - p^{-(\alpha-2)})^2}{p^{2-\alpha}(p^{\alpha-1} - 1)^2} p^{-(\alpha-2)k} (1 + o(1)).$$

For $\alpha = 2$ we similarly have

$$b_k < \frac{(1 - p^{-2})(1 - p^{-1})^2}{(p - 1)^2} k^{-2} (1 + o(1)).$$

In the case $1 < \alpha < 2$ we have

$$b_k < \frac{p^\alpha - 1}{(p^{\alpha-1} - 1)^2} \left[\frac{p^{-\alpha} - p^{-\alpha(k+1)}}{(1 - p^{-\alpha})(p^{-k-1} - p^{-1})^2} + \frac{p^{-\alpha(k+2)}}{(1 - p^{-\alpha})(p^{-k-1})^2} + \frac{p^{2-\alpha}p^{-(\alpha-2)k}}{\delta_{\min}^2} \right]^{-1} (1 + o(1)),$$

which gives

$$b_k < \frac{(p^\alpha - 1)\delta_{\min}^2}{p^{2-\alpha}(p^{\alpha-1} - 1)^2} p^{-(2-\alpha)k} (1 + o(1)).$$

Let us now estimate b_k from below. For $\alpha > 2$, we have

$$b_k > \frac{p^\alpha - 1}{(p^{\alpha-1} - 1)^2} \left[\sum_{i=1}^k \frac{p^{-\alpha i}}{(p^{-k-1} - p^{-i})^2} + \sum_{i=k+2}^{\infty} \frac{p^{-\alpha i}}{(p^{-k-1} - p^{-i})^2} + \frac{p^{2-\alpha}p^{-(\alpha-2)k}}{(1 - p^{-1})^2(1 - p^{-\alpha+2})^2} \right]^{-1} (1 + o(1)),$$

which gives

$$\frac{(p^\alpha - 1)(1 - p^{-1})^2(1 - p^{-(\alpha-2)})^2}{p^{2-\alpha}(p^{\alpha-1} - 1)^2} p^{-(\alpha-2)k} (1 + o(1)) < b_k.$$

In the case $\alpha = 2$, we have

$$b_k > \frac{(p^2 - 1)}{(p - 1)^2} k^{-2} (1 + o(1)).$$

For $1 < \alpha < 2$, we get

$$b_k > \frac{p^\alpha - 1}{(p^{\alpha-1} - 1)^2} \left[\frac{p^{-\alpha} - p^{-\alpha(k+1)}}{(1 - p^{-\alpha})(p^{-k-1} - p^{-k})^2} + \frac{p^{-\alpha k}p^{-2\alpha}p^{-\alpha i}}{(1 - p^{-\alpha})(p^{-k-1} - p^{-k-2})^2} + \frac{p^{2-\alpha}p^{-(\alpha-2)k}}{\delta_{\max}^2} \right]^{-1} (1 + o(1)),$$

and hence,

$$b_k > \frac{(p^\alpha - 1) \delta_{\max}^2}{p^{2-\alpha} (p^{\alpha-1} - 1)^2} p^{-(2-\alpha)k} (1 + o(1)).$$

Collecting the above inequality, we write the lower and upper estimates for b_k as $k \rightarrow \infty$ for different α as follows:

$$a(\alpha, k) (1 + o(1)) < b_k < b(\alpha, k) (1 + o(1)), \quad (55)$$

where

$$a(\alpha, k) = (1 + o(1)) \begin{cases} \frac{(p^\alpha - 1) (1 - p^{-1})^2 (1 - p^{-\alpha+2})^2}{p^{2-\alpha} (p^{\alpha-1} - 1)^2} p^{-(\alpha-2)k}, & \alpha > 2, \\ \frac{(p^2 - 1)}{(p - 1)^2} k^{-2}, & \alpha = 2, \\ \frac{(p^\alpha - 1) \delta_{\max}^2}{p^{2-\alpha} (p^{\alpha-1} - 1)^2} p^{-(2-\alpha)k}, & 1 < \alpha < 2, \end{cases}$$

$$b(\alpha, k) = (1 + o(1)) \begin{cases} \frac{(p^\alpha - 1) (1 - p^{-\alpha+2})^2}{p^{2-\alpha} (p^{\alpha-1} - 1)^2} p^{-(\alpha-2)k}, & \alpha > 2, \\ \frac{(1 - p^{-2}) (1 - p^{-1})^2}{(p - 1)^2} k^{-2}, & \alpha = 2, \\ \frac{(p^\alpha - 1) \delta_{\min}^2}{p^{2-\alpha} (p^{\alpha-1} - 1)^2} p^{-(2-\alpha)k}, & 1 < \alpha < 2. \end{cases}$$

Appendix D. Asymptotic estimate of the series

Here, we shall obtain the estimate for the series

$$S_1(t) = \sum_{i=1}^{\infty} i^{-k} a^{-i} \exp(-b^{-i}t), \quad t \geq 0, \quad a > 1, \quad b > 1, \quad k \geq 0 \quad (56)$$

for large t .

Lemma 4. *For series (56) we have the following asymptotic estimate as $t \rightarrow \infty$:*

$$a^{-1} (\ln b)^{k-1} t^{-\frac{\ln a}{\ln b}} (\ln t)^{-k} \Gamma\left(\frac{\ln a}{\ln b}\right) (1 + o(1)) \leq S_1(t) \leq a (\ln b)^{k-1} t^{-\frac{\ln a}{\ln b}} (\ln t)^{-k} \Gamma\left(\frac{\ln a}{\ln b}\right) (1 + o(1)),$$

where $\Gamma(z)$ is the Gamma function and $o(1)$ is an infinitesimal quantity as $t \rightarrow \infty$.

Proof. Instead of series (56), we consider the series

$$S_2(t) = \sum_{i=2}^{\infty} i^{-k} a^{-i} \exp(-b^{-i}t) = S_1(t) - a^{-1} \exp(-b^{-1}t).$$

We note that $\frac{1}{x^k} a^{-x}$ is a decreasing function and $\exp(-b^{-x}t)$ is an increasing function of x . Hence, on the interval $i \leq x \leq i+1$ we have

$$\frac{1}{x^k} a^{-x} \exp(-b^{-(x-1)}t) \leq a^{-i} \exp(-b^{-i}t) \leq \frac{1}{(x-1)^k} a^{-(x-1)} \exp(-b^{-x}t). \quad (57)$$

Integrating (57) in x from i to $i+1$, we find that

$$a^{-1} \int_i^{i+1} \frac{1}{x^k} a^{-(x-1)} \exp(-b^{-(x-1)}t) dx \leq a^{-i} \exp(-b^{-i}t) \leq a \int_i^{i+1} \frac{1}{(x-1)^k} a^{-x} \exp(-b^{-x}t) dx. \quad (58)$$

Next, summing (58) in i from 2 to ∞ , this gives

$$S_{\min}(t) \leq S_2(t) \leq S_{\max}(t),$$

where

$$S_{\min}(t) = a^{-1} \int_2^{\infty} \frac{1}{x^k} a^{-(x-1)} \exp(-b^{-(x-1)}t) dx, \quad (59)$$

$$S_{\max}(t) = a \int_2^{\infty} \frac{1}{(x-1)^k} a^{-x} \exp(-b^{-x}t) dx. \quad (60)$$

In (59) and (60) we change the variable of integration as $b^{-(x-1)}t = y$ and $b^{-x}t = y$, respectively. We have

$$S_{\min}(t) = a^{-1} (\ln b)^{-1} t^{-\frac{\ln a}{\ln b}} \int_0^{b^{-1}t} \left(\frac{\ln t - \ln y}{\ln b} + 1 \right)^{-k} y^{\frac{\ln a}{\ln b}-1} \exp(-y) dy,$$

$$S_{\max}(t) = a (\ln b)^{-1} t^{-\frac{\ln a}{\ln b}} \int_0^{b^{-2}t} \left(\frac{\ln t - \ln y}{\ln b} - 1 \right)^{-k} y^{\frac{\ln a}{\ln b}-1} \exp(-y) dy.$$

If $k > 0$, then, as $t \rightarrow \infty$,

$$S_{\min}(t) = a^{-1} (\ln b)^{k-1} t^{-\frac{\ln a}{\ln b}} (\ln t)^{-k} \Gamma\left(\frac{\ln a}{\ln b}\right) (1 + o(1)),$$

$$S_{\max}(t) = a (\ln b)^{k-1} t^{-\frac{\ln a}{\ln b}} (\ln t)^{-k} \Gamma\left(\frac{\ln a}{\ln b}\right) (1 + o(1)),$$

which gives the assertion of the lemma. \square

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